Carleton UNIVERSITY
Canada's Capital University

Mathematics Tutorial Series
Differential Calculus #12

Exponential Functions

An exponential function has a formula like:

$$y = r^x$$

for some real number $r > 0$ called the base of the exponential.

The variable is in the exponent.

The basic property of exponential functions is:

$$r^{a+b} = r^a r^b$$

There are a few common choices for base.

$r = 10$ is used in the definition of "decimal" numbers, in the metric system, pH calculations, scientific notation

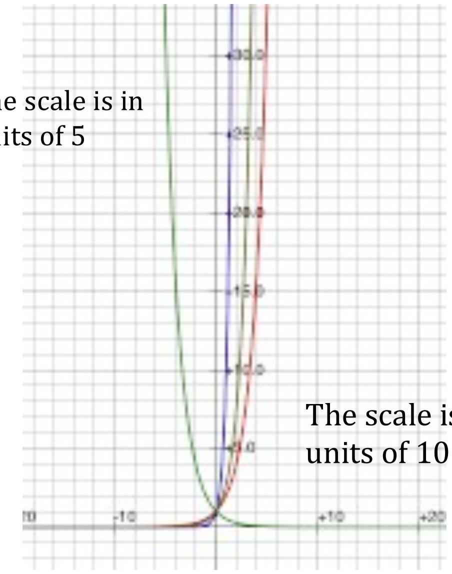
$r = 2$ is used in the binary system, for measuring computer files, for algorithm analysis, in the digital world

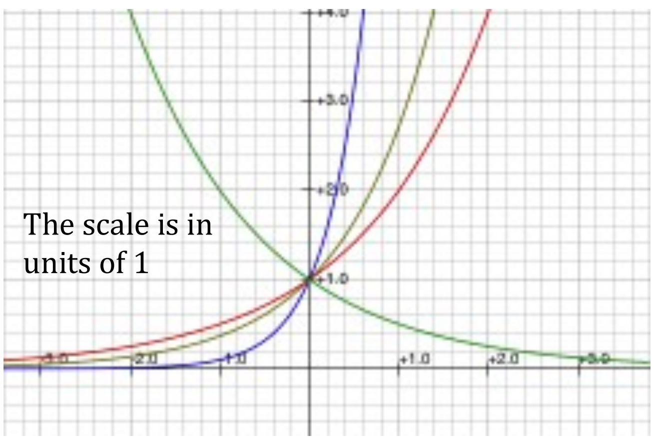
$r = \frac{1}{2}$ is used for radioactive models, half-life models

r = 1.03 is used to model a 3% interest rate.

$r = e =$ 2.7182818285 … is a magic value that is used everywhere inside and outside mathematics and its applications.

Graphs of $y = 10^x, 2^x, 0.5^x, e^x$





Notation

e^x is also commonly written as $exp(x)$.
You will see it this way on some calculators.

What does "exponential growth" mean?

Popular culture: "very fast"

Mathematical: grows in proportion to current value. Changes faster than any polynomial.

$y = r^x$ with $0 < r < 1$ is a **decreasing** function.

Derivative

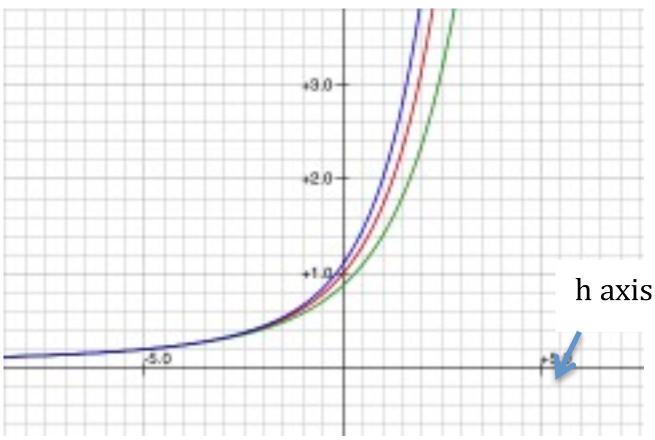
Use the formal definition.

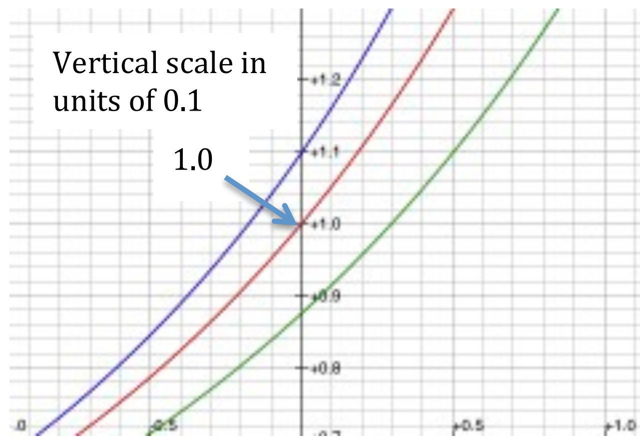
Suppose $y = r^x$. Then

$$y'(a) = \lim_{h \to 0} \frac{r^{a+h} - r^a}{h}$$
$$= \left(\lim_{h \to 0} \frac{r^h - 1}{h} \right) r^a$$

For each value of r, this limit is some number that doesn't change with the number a.

Look at some graphs of $y = \frac{r^h - 1}{h}$ for
$\boldsymbol{r = 3}$ (blue), $\boldsymbol{r = 2.7}$ (red), $\boldsymbol{r = 2.5}$ (green)
This formula is invalid at $h = 0$.





The number e is defined as the real number for which $\left(\lim_{h \to 0} \frac{r^h - 1}{h} \right) = 1$.

With this definition we get:

$$(e^x)' = \left(\lim_{h \to 0} \frac{e^h - 1}{h} \right) e^x = e^x$$

The exponential function with base e is equal to its derivative.

The decimal expresion for e does not end or have a repeating pattern.
It is a **transcental number**. i.e. satisfies no algebraic equation.

$$e = 2.718281828459045235360287471 \ldots$$

Summary:

The function $e^x = \exp(x)$ is defined for all real numbers x. Also $exp(0) = 1$ and $exp(x) > 0$ for all x.

$$\frac{de^x}{dx} = (e^x)' = e^x$$

$$\lim_{x \to +\infty} e^x = +\infty$$

$$\lim_{x \to -\infty} e^x = 0$$

Exercises: Calculate the derivatives

$(12e^x)'$

$(12e^{3x})'$

$\left(e^{x^2}\right)'$

$\left(e^{\sqrt{x}}\right)'$

$\left(xe^{x^2}\right)'$

$\left(e^{\sin x}\right)'$

$(e^x \tan x)'$

DON'T LOOK AT THE ANSWERS
UNTIL YOU HAVE TRIED TO SOLVE THE PROBLEMS

Answers:

$(12e^x)' = 12\, e^x$

$(12e^{3x})' = 12e^{3x}(3x)' = 12\mathrm{e}^{3\mathrm{x}}3 = 36\mathrm{e}^{3\mathrm{x}}$

$\left(e^{x^2}\right)' = e^{x^2}(x^2)' = e^{x^2}(2x) = 2xe^{x^2}$

$\left(e^{\sqrt{x}}\right)' = e^{\sqrt{x}}\left(\sqrt{x}\right)' = e^{\sqrt{x}}\,\frac{1}{2\sqrt{x}}$

$\left(xe^{x^2}\right)' = (x)'e^{x^2} + x\left(e^{x^2}\right)' = e^{x^2} + x\left(2xe^{x^2}\right) = (1 + 2x^2)\, e^{x^2}$

$\left(e^{\sin x}\right)' = e^{\sin x}(\sin x)' = (\cos x)e^{\sin x}$

$(e^x \tan x)' = \sec^2 x\, e^x + \tan x\, e^x = (\sec^2 x + tanx)\, e^x$